Tools in the orbit space approach to the study of invariant functions: rational parametrization of strata

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Abstract
Functions which are equivariant or invariant under the transformations of a
compact linear group G acting in a Euclidean space $\mathbb{R}^n$, can profitably be
studied as functions defined in the orbit space of the group. The orbit space
is the union of a finite set of strata, which are semialgebraic manifolds formed
by the G-orbits with the same orbit-type. In this paper, we provide a simple
recipe to obtain rational parametrizations of the strata. Our results can be
easily exploited, in many physical contexts where the study of equivariant or
invariant functions is important, for instance in the determination of patterns
of spontaneous symmetry breaking, in the analysis of phase spaces and
structural phase transitions (Landau theory), in equivariant bifurcation theory,
in crystal field theory and in most areas where use is made of symmetry-adapted
functions.

  A physically significant example of utilization of the recipe is given,
related to spontaneous polarization in chiral biaxial liquid crystals, where the
advantages with respect to previous heuristic approaches are shown.

1. Introduction

The determination of properties of functions which are equivariant or invariant under the
transformations of a compact linear group (hereafter abbreviated as CLG) G is often a basic
problem to solve in many physical contexts. G-invariant functions play an important role, for
instance, in the determination of patterns of spontaneous symmetry breaking and structural
phase transitions (Landau theory [1–4]), in equivariant bifurcation theory (see, for instance,
[5] and references therein), in crystal field theory and in most areas of solid state theory.

  It will not be essentially restrictive, in the following, to assume that G is a matrix subgroup
of the real group O(n). In fact, complex linear groups can always be transformed into real

linear groups through a process of realification and compact real linear groups are equivalent to orthogonal linear groups.

An approach to the study of invariant functions that fully exploits invariance and possible regularity properties, takes advantage of the fact that a G-invariant function f, defined on $\mathbb{R}^n$, is a constant along each orbit of G and can, therefore, be considered as a function $\widehat{f}$ in the orbit space $\mathbb{R}^n/G$ of the action of G in $\mathbb{R}^n$. Geometric-invariant theory [6, 7] suggests how to get, in principle, $\widehat{f}$ and $\mathbb{R}^n/G$.

The orbit space of a compact group can be realized as a connected semialgebraic subset (i.e., a subset defined by algebraic equalities and inequalities) of a Euclidean space $\mathbb{R}^q$. It turns out to be formed by the union of connected semialgebraic manifolds of different dimensions (*primary strata*). The images of G-orbits with the same orbit-type form *isotropy-type strata*, whose connected components are primary strata.

The properties of a function defined on $\mathbb{R}^n/G$, may critically depend on the geometry of this space, that has to be known explicitly. A simple way to obtain a substantial determination of the algebraic equations and inequalities defining the range of an orbit map and its strata has been suggested in [8–10] (see also [11, 12]). These relations can be expressed in the form of positivity and rank conditions of a matrix $\widehat{P}(p)$, whose elements are polynomial functions of $p \in \mathbb{R}^q$. In this way, the defining relations of the strata are obtained in the form of algebraic equations and inequalities.

In applications or, simply, to get a better understanding of the geometry of the stratum (its connectivity properties, its boundary, etc), one often needs to solve explicitly these relations and, sometimes, this is very difficult to do using standard algorithms. In this paper, we propose a general method to derive the equations defining the stratification of orbit spaces of CLGs, in the form of explicit rational parametric relations.

This paper is organized as follows. In section 2, we summarize the results in the P-matrix approach to the characterization of the orbit spaces of CLGs [8, 9]. In section 3, we present and prove our results, and in section 4, we illustrate them in a physically significant example, related to spontaneous polarization in chiral biaxial liquid crystals.

2. An overview of the geometry of linear group actions

In this section, we shall recall, without proofs, some results concerning invariant theory and the geometry of orbit spaces of CLGs (see, for instance, [13–15] and references therein).

For our purposes, it will not be restrictive to assume that G is a matrix subgroup of the group O(n), acting linearly in the Euclidean space $\mathbb{R}^n$ and thus defining the G-space $(G, \mathbb{R}^n)$.

We shall denote by $g \cdot x$ the action of $g \in G$ on $x \in \mathbb{R}^n$ and by G_x the isotropy subgroup of G at $x \in \mathbb{R}^n$. The isotropy subgroups of G at points of the orbit $G \cdot x$ form the conjugacy class (G_x) of G_x in G, which identifies the *orbit-type* of (the points of) $G \cdot x$. The set of points $x \in \mathbb{R}^n$ (of G-orbits) with the same orbit-type forms an *isotropy-type stratum of* $(G, \mathbb{R}^n)$.

The *orbit space* of the action of G in $\mathbb{R}^n$ is defined as the quotient space $\mathbb{R}^n/G$, endowed with the quotient topology and differentiable structure. The image of a stratum of $\mathbb{R}^n$, through the canonical projection $\pi : \mathbb{R}^n \to \mathbb{R}^n/G$, defines an *isotropy-type stratum* of $\mathbb{R}^n/G$; the connected components of an *isotropy-type stratum* of $\mathbb{R}^n/G$ are iso-dimensional manifolds (*primary strata of* $\mathbb{R}^n/G$), but the orbit space is not a manifold.

Almost all the points of $\mathbb{R}^n/G$ belong to a unique stratum, the *principal stratum*, which is a connected open dense subset of $\mathbb{R}^n/G$. The boundary of the principal stratum is the union of disjoint *singular* strata. Every stratum $\widehat{S}$ of $\mathbb{R}^n/G$ is open in its topological closure $\overline{\widehat{S}}$.

The following partial ordering can be introduced in the set of all the orbit-types: $(H) < (K)$, if H is conjugate with a subgroup of K. If (H_i) is the orbit-type of the stratum $\widehat{S}_i$

of $\mathbb{R}^n/G$, $i = 1, 2$, then $H_1 < H_2$, iff $\widehat{S}_2$ is in the boundary of $\widehat{S}_1$; therefore, more peripheral strata of $\mathbb{R}^n/G$ have greater orbit-type. The number of distinct orbit-types of G is finite and there is a unique minimum orbit-type, the *principal orbit-type*, corresponding to the principal stratum.

The ring $\mathbb{R}[x]^G$ of real G-invariant polynomial functions of x is finitely generated [16–19]. There exists, therefore, a finite minimal collection of homogeneous G-invariant polynomials $p(x) = (p_1(x), p_2(x), \ldots, p_q(x))$ (minimal integrity basis for the ring of G-invariant polynomials, henceforth abbreviated as MIB) such that any element $F \in \mathbb{R}[x]^G$ can be expressed as a polynomial function $\widehat{F}$ of $p(x)$:

$$\widehat{F}(p(x)) = F(x) \qquad \forall x \in \mathbb{R}^n. \tag{1}$$

The number q of elements of a minimal integrity basis and their homogeneity *degrees* d_i are only determined by the group G.

The group G is said to be *coregular* if the elements of its MIBs are algebraically independent. The elements of a MIB of a non-coregular group satisfy a set of algebraic identities in $\mathbb{R}^n$: $\widehat{F}_A(p(x)) = 0$, $A = 1, \ldots$ and the associated set of equations

$$\widehat{F}_A(p) = 0 \qquad A = 1, \ldots \tag{2}$$

defines an algebraic variety in $\mathbb{R}^q$, which is called the variety $\mathcal{Z}$ of the relations (among the elements of the MIB). If G is coregular, $\mathcal{Z} = \mathbb{R}^q$.

Since G is a compact group, the orbits of G are separated by the elements of a MIB of G (for an elementary proof see, for instance, [9]), i.e., at least one element of a MIB of G takes on different values at two distinct orbits. Thus, the elements of a MIB of G yield a parametrization of the points of $\mathbb{R}^n/G$, that turns out to be also smooth, since the *orbit map* $p : \mathbb{R}^n \longrightarrow \mathbb{R}^q$, which maps all the points of $\mathbb{R}^n$ lying on an orbit of G onto a single point of $\mathbb{R}^q$, induces a diffeomorphism of $\mathbb{R}^n/G$ onto a semialgebraic q-dimensional connected closed subset of $\mathbb{R}^q$. Like all semialgebraic varieties [20], $p(\mathbb{R}^n)$ presents a natural stratification in connected semialgebraic sub-varieties, called *primary strata*, which turn out to be the connected components of the isotropy-type strata[1].

A characterization of the image $p(\mathbb{R}^n)$ of the orbit space of G as a semi-algebraic variety and of its primary strata can be easily obtained through a matrix $\widehat{P}(p)$, defined only in terms of the G-invariant Euclidean scalar products between the gradients of the elements of the MIB $\{p(x)\}$:

$$P_{ab}(x) = \sum_{i=1}^{n} \partial_i p_a(x) \partial_i p_b(x) = \widehat{P}_{ab}(p(x)) \qquad a, b = 1, \ldots, q \tag{3}$$

where in the last member, use has been made of Hilbert's theorem, in order to express $P_{ab}(x)$ as a polynomial function of $p_1(x), \ldots, p_q(x)$.

The following theorem [8–10] (see also [11]) clarifies the meaning and points out the role of the matrix $\widehat{P}(p)$.

Theorem 2.1. *Let G be a compact subgroup of the real group $O(n)$, p the orbit map $\mathbb{R}^n \to \mathbb{R}^q$ defined by the MIB $(p_1(x), p_2(x), \ldots, p_q(x))$ and $\widehat{P}(p)$ the matrix defined in (3). Then the range $p(\mathbb{R}^n)$ of p is the unique semialgebraic connected subset of the variety $\mathcal{Z} \subseteq \mathbb{R}^q$ of the relations among the elements of the MIB where $\widehat{P}(p)$ is positive semi-definite. The k-dimensional primary strata of $p(\mathbb{R}^n)$ are the connected components of the set*

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[1] A simple example of a compact connected *linear* semialgebraic variety of $\mathbb{R}^3$ is yielded by a polyhedron: its interior points form a unique three-dimensional primary stratum (principal stratum), while two-, one- and zero-dimensional primary strata are formed, respectively, by the interior points of each face, by the interior points of each edge and by each vertex.

$\{p \in \mathcal{Z} \mid \widehat{P}(p) \geqslant 0, \operatorname{rank}(\widehat{P}(p)) = k\}$; *they are the images of the connected components of the k-dimensional isotropy-type strata of $\mathbb{R}^n/G$. In particular, the set of interior points of $p(\mathbb{R}^n)$, where $\widehat{P}(p)$ has the maximum rank, is the image of the principal stratum.*

In the following, we shall identify orbit spaces and their strata with their images through orbit maps.

If G is coregular, the p_i, with range in the semialgebraic set Δ defined by the inequalities $\widehat{P}(p) > 0$ (and no equation!), provide a one-to-one parametrization of the principal stratum of $\mathbb{R}^n/G$. If the stratum is singular, the p_i are not independent parameters, being bounded also by the equations defining the stratum. In applications or, simply, to get a better understanding of the geometry of the stratum (its connectivity properties, its boundary, etc), one often needs to solve these equations explicitly, which are obtained in implicit form from the theorem recalled above. Often, this is very difficult to do using standard algorithms. In the following section, we propose a general method to derive the equations defining the singular strata of the orbit space of any CLG in the form of explicit rational parametric relations.

The mathematical apparatus just recalled offers effective tools, for instance, in the determination of possible patterns of spontaneous symmetry breaking, when the ground state of the system is determined by the absolute minimum of an invariant potential. Let us recall the basic ideas. In this context, the vector $x \in \mathbb{R}^n$ is an order parameter and G is the symmetry group of the potential $\Phi(\alpha; x)$ (free energy, or Higgs potential, for instance), expressed in terms, also, of parameters α. The points $x_0(\alpha)$, where the function $\phi_\alpha(x) = \Phi(\alpha; x)$ takes on its absolute minimum, determine the stable phase of the system, whose residual symmetry is defined by the isotropy subgroup of G at x_0. Owing to the G-invariance of the potential, each of its stationary points is degenerate along the G-orbit through it. Since the isotropy subgroups of G at points of the same orbit are conjugate in G, only the orbit-type of $x_0(\alpha)$ is physically relevant. *Structural phase transitions* take place when, by varying the values of the α, the point $x_0(\alpha)$ is shifted to a different stratum.

If $\Phi(\alpha; x)$ is a sufficiently general function of the α, by varying these parameters, the point $x_0(\alpha)$ can be shifted to any stratum of $\mathbb{R}^n/G$. So, *the strata are in a one-to-one correspondence with the symmetry phases allowed by the G-invariance of the potential.* In contrast, extra restrictions on the form of the potential function, not coming from G-symmetry requirements (e.g., the assumption that the potential is a polynomial of low degree), can limit the number of allowed structural phases for the system in its ground state.

3. Parametrization of strata in orbit spaces of compact linear groups

Let G be a compact linear group, acting orthogonally in $\mathbb{R}^n$, and $\{p\}$ be a related MIB. We shall prove that the possibility of parametrizing the principal stratum of the orbit space of a coregular group in terms of parameters related to the elements of a MIB can be extended to singular strata. The result stems from the proof of the following statement: *the set of interior points of the topological closure of an isotropy-type stratum, with orbit-type (H), is diffeomorphic to the principal stratum of the orbit space of a group space of the stabilizer of H in G.* If the stabilizer turns out to be coregular, the constructive proof of this proposition will provide a one-to-one rational parametrization of the stratum. Since, in view of possible applications, we are mainly interested in the possibility of getting a one-to-one parametrization of a stratum, in the presentation of our results, stress will be laid on this aspect of the matter.

Let H be a proper isotropy subgroup of the G-space $(G, \mathbb{R}^n)$. We shall denote by S the stratum of $(G, \mathbb{R}^n)$ with orbit-type (H) and by $\widehat{S}$ its image $p(S)$ in the orbit space $\mathbb{R}^n/G$. Let us also define

$$V = \{x \in \mathbb{R}^n \mid G_x \supseteq H\} \qquad \mathcal{V} = \{x \in \mathbb{R}^n \mid G_x = H\}. \tag{4}$$

The non-trivial sets V and $\mathcal{V}$ have the following properties, which are more or less immediate consequences of their definitions:

(i) $V = \{x \in \mathbb{R}^n \mid h \cdot x = x, h \in H\}$ is a linear subspace of $\mathbb{R}^n$, let us call ν its dimensions;

(ii) V is the topological closure of $\mathcal{V}$: $V = \overline{\mathcal{V}}$;

(iii) $\mathcal{V} = S \cap V$ and every G-orbit lying in S has at least a point in $\mathcal{V}$ so that $S = G \cdot \mathcal{V}$, $\overline{S} = G \cdot V$ (where the bar denotes topological closure) and, consequently,

$$p(\mathcal{V}) = \widehat{S} \qquad p(V) = \overline{\widehat{S}} \tag{5}$$

and

$$\mathrm{rank}(P(x)|_{x \in \mathcal{V}}) = \dim(\widehat{S}) \qquad \mathrm{rank}(P(x)|_{x \in V \setminus \mathcal{V}}) < \dim(\widehat{S}). \tag{6}$$

A G-orbit of S may intersect $\mathcal{V}$ at one or more points. In the first case, every G-orbit of S intersects $\mathcal{V}$ at a point and, owing to item (iii), the coordinates of the points of $\mathcal{V}$ provide a one-to-one rational (in effect, linear) parametrization of the orbits of G lying in S, obtained by solving the system of linear equations $h \cdot x = x \; \forall h \in H$. The allowed range of x, $x \in \mathcal{V}$ is determined by the inequalities assuring that $\mathrm{rank}(P(x)) = \dim(\widehat{S})$. The parameters x, with range $\mathcal{V}$, would provide, in this case, a one-to-one parametrization of the stratum, thus solving our problem. In general, however, the intersection of a G-orbit, of orbit-type (H), with the stratum S does not reduce to a single point. So, a sounder analysis is required, that will be the object of the rest of this section[2].

Two distinct points, x and $g \cdot x$, $g \in G$, of the same G-orbit, lie in $\mathcal{V}$ iff $h \cdot g \cdot x = g \cdot x \; \forall h \in H$, that is, iff g is in the stabilizer $\mathrm{Stab}(H, G)$ of H in G. The intersection of a G-orbit of S with $\mathcal{V}$ is, therefore, the $\mathrm{Stab}(H, G)$-orbit through x and $\mathrm{Stab}(H, G)$ is the largest subgroup of G leaving V invariant.

In the group space $(\mathrm{Stab}(H, G), V)$, the isotropy subgroup at a point of general position is H. Therefore, the principal stratum Σ satisfies the following relation:

$$V \supset \Sigma \supseteq \mathcal{V}. \tag{7}$$

It has to be noted that Σ could contain $\mathcal{V}$ in a strict sense, since, at points of $V \setminus \mathcal{V}$, the conjugacy class in G of the isotropy subgroup of $\mathrm{Stab}(H, G)$ could be smaller than the conjugacy class in G of the isotropy subgroup of G.

Now, H is an invariant subgroup of $\mathrm{Stab}(H, G)$ and a subgroup of all the isotropy subgroups of $(\mathrm{Stab}(H, G), V)$. Thus, the action of $\mathrm{Stab}(H, G)$ in V defines a linear group K (and the group space (K, V)), isomorphic to the quotient group $\mathrm{Stab}(H, G)/H$, through the relation

$$(sH) \cdot v = s \cdot v \qquad s \in \mathrm{Stab}(H, G) \quad sH \in \mathrm{Stab}(H, G)/H. \tag{8}$$

So, we can conclude that *the intersection of a G-orbit of S with $\mathcal{V}$ is an orbit of (K, V).* Obviously, the orbit spaces V/K and $V/\mathrm{Stab}(H, G)$ are isomorphic and can be identified.

We can rephrase the result just obtained, by claiming that the points of $\widehat{S}$ are in a one-to-one correspondence with the points of a, possibly proper, subset $\widehat{\mathcal{V}}$ of the principal stratum $\widehat{\Sigma}$ of the orbit space V/K and the following relations hold true:

$$\overline{\widehat{S}} = p(V) \supset p(\Sigma) \supseteq p(\mathcal{V}) = \widehat{S}. \tag{9}$$

Since, as stressed, we know how to parametrize a principal stratum, our problem is reduced to the determination of $\widehat{\mathcal{V}}$. We shall show that this can be easily done, making use of (6).

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[2] Despite its being, generally, not one-to-one, the parametrization of the orbits of a stratum by means of the points of V may be useful. It has been used, for instance, by Kim [21] to parametrize the strata of a set of low-dimensional orbit spaces of compact linear Lie groups.

To make the solution of the problem simpler, let us assume that an orthonormal basis has been introduced in $\mathbb{R}^n$ such that the first ν elements of the basis yield a basis for the vector space V. Then, if we denote by $V_\perp$ the orthogonal complement of V in $\mathbb{R}^n$, the subspace $V_\perp$ is invariant under (K and) H, owing to the orthogonality of the transformations of G. Since V contains all the H-invariant vectors of $\mathbb{R}^n$, there is no non-trivial H-invariant vector in $V_\perp$.

To attain our goal, we shall start from a convenient parametrization of the principal stratum $\widehat{\Sigma}$ of V/K in terms of l real parameters λ, related to a MIB $(\lambda_1(v), \ldots, \lambda_l(v))$ of the ring of polynomial K-invariant functions of $v \in V$. This parametrization will be global and one-to-one, if K turns out to be coregular. In this case, the range of λ has to be restricted to the positivity region of the $\widehat{P}$-matrix $\widehat{\Lambda}(\lambda)$, associated with the MIB $\{\lambda\}$:

$$\widehat{\Lambda}_{\alpha\beta}(\lambda(v)) = \Lambda_{\alpha\beta}(v) = \sum_{i=1}^{\nu} \partial_i \lambda_\alpha(x_1, \ldots, x_\nu) \partial_i \lambda_\beta(x_1, \ldots, x_\nu). \tag{10}$$

As recalled in the previous section, in fact, the orbit space V/K and its principal stratum $\widehat{\Sigma}$ can be identified, respectively, with the semialgebraic sets $\lambda(V)$ and $\lambda(\Sigma)$:

$$\widehat{\Sigma} = \lambda(\Sigma) = \{\lambda \in \mathbb{R}^l \mid \widehat{\Lambda}(\lambda) > 0\} \qquad V/K = \overline{\widehat{\Sigma}} = \lambda(V) = \{\lambda \in \mathbb{R}^l \mid \widehat{\Lambda}(\lambda) \geqslant 0\} \tag{11}$$

the second set being the closure of the first one. Moreover, the definition of $\widehat{\mathcal{V}}$ and equation (7) imply

$$\widehat{\mathcal{V}} = \lambda(\mathcal{V}) \subseteq \lambda(\Sigma) = \widehat{\Sigma}. \tag{12}$$

If K is not coregular, only a local one-to-one parametrization can be obtained for $\widehat{\Sigma}$, by eliminating redundant elements in the set of parameters $(\lambda_1, \ldots, \lambda_l)$, through the solution of the algebraic relation(s) among the elements of the MIB $\{\lambda\}$, and imposing convenient semi-positivity and rank conditions on the matrix $\widehat{\Lambda}(\lambda)$.

A one-to-one local or global parametrization of $\widehat{\Sigma}$ yields, obviously, also a local or global one-to-one parametrization of $\widehat{\mathcal{V}}$, provided that additional restrictions are imposed on the range of λ, whenever $\widehat{\mathcal{V}}$ is a proper subset of $\widehat{\Sigma}$. In this case, the correct bounds can be obtained in the following way (we shall only consider the case of a coregular K, the extension of the results to non-coregular K is straightforward, but, as just stressed, may lead to a loss of globality).

When x spans V, the elements of the MIB $(p_1(x), \ldots, p_q(x))$ of G define a set of K-invariant polynomial functions of $v = (x_1, \ldots, x_\nu) \in V$. Therefore, by the Hilbert theorem recalled in the previous section, they can be expressed as polynomial functions of λ, that is,

$$p|_V = \phi \circ \lambda. \tag{13}$$

Let us remark that, in our assumptions, there are no relations among the elements of the MIB $\{\lambda\}$ and, consequently, possible relations $F_\alpha(p) = 0$ among the p_i are identically satisfied for $p = \phi(\lambda)$.

From (13) and (7), one immediately obtains

$$\overline{\widehat{S}} = p(V) = \phi(V/K) \supset \phi(\widehat{\Sigma}) \supseteq \phi(\widehat{\mathcal{V}}) = p(\mathcal{V}) = \widehat{S} \tag{14}$$

and, since $\widehat{\Sigma}$, being a principal stratum, is the set of interior points of V/K and is connected, $\phi(\widehat{\Sigma})$ will coincide with the set of interior points of the closure $\overline{\widehat{S}}$ of $\widehat{S}$ and will be connected. This set does not coincide with $\widehat{S}$ if $\overline{\widehat{S}}$ contains, in its interior, points representing bordering strata of $\widehat{S}$. This certainly happens if the set $\widehat{S}$ is not connected and, presumebly, also if it is not multiply connected[3].

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[3] To our knowledge, no general proof exists even of simple connectivity of the principal stratum of a coregular compact group; however, principal strata of coregular low-dimensional ($D \leqslant 4$) orbit spaces can be checked to be multiply connected [22, 23].

The identification of points $\lambda \in \widehat{\Sigma}$, if any, whose image $\phi(\lambda) \notin \widehat{S}$, can be obtained in the following way.

Let $x = v \oplus v_\perp$ yield the decomposition of $x \in \mathbb{R}^n$ in its vector components $v \in V$ and $v_\perp \in V_\perp$: $v = (x_1, \ldots, x_\nu)$, $v_\perp = (x_{\nu+1}, \ldots, x_n)$. Then, a G-invariant polynomial $f(x)$ can be thought of as a polynomial in v and $v_\perp$ and it is easy to realize that it cannot contain linear terms in $v_\perp$, being v invariant under H. As a consequence,

$$\partial_i f(x) = 0 \qquad \text{for} \quad x \in V \quad \text{and} \quad i = \nu + 1, \ldots, n. \tag{15}$$

Now, starting from the very definition of $P(x)$ (see (3)), for every $x = v \in V$ we obtain, using (15) and the identity $p(v) = \phi(\lambda(v))$,

$$\begin{aligned}
\widehat{P}_{ab}(p(v)) &= \sum_{i=1}^{\nu} \partial_i p_a(v) \partial_i p_b(v) \\
&= \sum_{\alpha,\beta=1}^{l} \partial_\alpha \phi_a(\lambda) \partial_\beta \phi_b(\lambda) \Big|_{\lambda=\lambda(v)} \sum_{i=1}^{\nu} \partial_i \lambda_\alpha(v) \partial_i \lambda_\beta(v) \\
&= \left(J(\lambda) \widehat{\Lambda}(\lambda) J(\lambda)^{\mathrm{T}} |_{\lambda=\lambda(v)} \right)_{ab}
\end{aligned} \tag{16}$$

where the superscript T denotes transposition and $J(\lambda)$ is the Jacobian matrix of the transformation $p = \phi(\lambda)$:

$$J_{a\alpha}(\lambda) = \partial_a \phi_\alpha(\lambda) \qquad a = 1, \ldots, q \quad \alpha = 1, \ldots, l. \tag{17}$$

So, we can conclude that, for all $\lambda \in V/K$,

$$\widehat{P}(\phi(\lambda)) = J(\lambda) \widehat{\Lambda}(\lambda) J(\lambda)^{\mathrm{T}}. \tag{18}$$

Equation (18) leads to an easy calculation of the points $\lambda \in \widehat{\mathcal{V}}$, that is, of the points $\lambda \in \widehat{\Sigma}$ whose image $p = \phi(\lambda)$ is in $\widehat{S}$. In fact, from (6), these points are characterized by the conditions $\lambda \in \widehat{\Sigma}$ and $\mathrm{rank}(\widehat{P}(\phi(\lambda))) = l$. Since, for $\lambda \in \widehat{\Sigma}$, the matrix $\widehat{\Lambda}(\lambda)$ is positive definite with rank l, we can conclude that the range of λ assuring the location of the point $p = \phi(\lambda)$ in $\widehat{S}$ is the semialgebraic subset Δ of $\mathbb{R}^l$, determined by the following inequalities:

$$\Delta = \{\lambda \in \mathbb{R}^l \mid \widehat{\Lambda}(\lambda) > 0 \text{ and } \mathrm{rank}(J(\lambda)) = l\}. \tag{19}$$

For $\lambda \in \Delta$, the relation $p = \phi(\lambda)$ yields a global rational parametrization for $\widehat{S}$.

It will be worthwhile to stress that the parametrization we have suggested is, in some way, canonical: the unique arbitrariness in the choice of the parameters is related to the choice of the MIBs.

An important byproduct of the result just proved is a simple test of the connection of $\widehat{S}$, that could be difficult or impossible to read directly from the equations of the stratum in implicit form. In fact, the condition that the boundary of $\overline{\widehat{S}}$ coincides with the boundary of $\widehat{S}$ is equivalent to the condition $\mathrm{rank}(J(\lambda)) = l$ for all $\lambda \in \widehat{\Sigma}$.

If K is not coregular, there are polynomial relations $F_A(\lambda) = 0$ among the elements of the MIB $(\lambda_1, \ldots, \lambda_l)$ and $l > \dim(\widehat{S})$. As already stressed, to obtain a one-to-one parametrization of the points of $\widehat{S}$ by means of the λ_α, one has to eliminate the redundant parameters by solving the equations $F_A(\lambda) = 0$. This may be feasible, but only locally and the resulting parametrization will not be global and possibly not rational.

4. An example

In this section, we shall show how the parametrization technique works in a simple example. The notation will be the same defined in the previous sections.

We shall consider the orthogonal linear group G defined by the action of the real group $O(3)$ in the real eight-dimensional space spanned by the independent components $x_1, \ldots, x_5$ of a symmetric and traceless tensor Q and the three components x_6, x_7, x_8 of a polar vector $\boldsymbol{P}$. To be specific, if O is a generic 3×3 real orthogonal matrix, the transformation rules of the x_i are obtained from the following relations:

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -\dfrac{2}{\sqrt{3}}x_1 & x_3 & x_4 \\[2mm] x_3 & \dfrac{x_1}{\sqrt{3}} - x_2 & x_5 \\[2mm] x_4 & x_5 & \dfrac{x_1}{\sqrt{3}} + x_2 \end{pmatrix} \tag{20}$$

$$Q'_{\alpha\beta} = \sum_{\gamma\delta=1}^{3} O_{\alpha\gamma} Q_{\gamma\delta} O_{\beta\delta} \qquad P'_{\alpha} = \sum_{\beta=1}^{3} O_{\alpha\beta} P_{\beta} \qquad \alpha, \beta = 1, 2, 3. \tag{21}$$

It is worthwhile to remark that the reflection $I = \mathrm{diag}(-1, -1, -1) \in O(3)$ reverses only the signs of the last three coordinates (x_6, x_7, x_8) and, due to the symmetry of the tensor Q, each G-orbit contains points where Q takes on a diagonal form ($x_3 = x_4 = x_5 = 0$). This remark makes much easier the determination of 'typical points' in strata, from which the orbit-type of the stratum can be identified [26, 21].

The real orthogonal linear group G, just defined, is coregular (see [27]) and its isotropy subgroup at a generic point of $\mathbb{R}^8$ is trivial. As a consequence, the principal orbits are three-dimensional manifolds, the principal stratum has dimensions five and there are five independent elements in a MIB (see, for instance, [15]), which can be chosen in the following way:

$$p_1 = \mathrm{Tr}\, Q^2 + \boldsymbol{P} \cdot \boldsymbol{P} = \sum_{j=1}^{8} x_j^2$$

$$p_2 = \boldsymbol{P} \cdot \boldsymbol{P} = \sum_{j=6}^{8} x_j^2$$

$$p_3 = 6\sqrt{2}\,\mathrm{Tr}\, Q^3 = -2\sqrt{3}x_1^3 + 6\sqrt{3}x_1x_2^2 - 3\sqrt{3}x_1x_3^2 - 9x_2x_3^2 - 3\sqrt{3}x_1x_4^2$$
$$\qquad\qquad + 9x_2x_4^2 + 18x_3x_4x_5 + 6\sqrt{3}x_1x_5^2$$

$$p_4 = 3\sqrt{2}\sum_{\alpha\beta} P_\alpha Q_{\alpha\beta} P_\beta = -2\sqrt{3}x_1x_6^2 + 6x_3x_6x_7 + \sqrt{3}x_1x_7^2 - 3x_2x_7^2 + 6x_4x_6x_8$$

$$\qquad\qquad + 6x_5x_7x_8 + \sqrt{3}x_1x_8^2 + 3x_2x_8^2 \tag{22}$$

$$p_5 = 6\sum_{\alpha\beta} P_\alpha Q_{\alpha\beta}^2 P_\beta = 4x_1^2x_6^2 + 3x_3^2x_6^2 + 3x_4^2x_6^2 - 2\sqrt{3}x_1x_3x_6x_7 - 6x_2x_3x_6x_7$$

$$\qquad\qquad + 6x_4x_5x_6x_7 + x_1^2x_7^2 - 2\sqrt{3}x_1x_2x_7^2 + 3x_2^2x_7^2 + 3x_3^2x_7^2 + 3x_5^2x_7^2$$
$$\qquad\qquad - 2\sqrt{3}x_1x_4x_6x_8 + 6x_2x_4x_6x_8 + 6x_3x_5x_6x_8 + 6x_3x_4x_7x_8 + 4\sqrt{3}x_1x_5x_7x_8$$
$$\qquad\qquad + x_1^2x_8^2 + 2\sqrt{3}x_1x_2x_8^2 + 3x_2^2x_8^2 + 3x_4^2x_8^2 + 3x_5^2x_8^2.$$

The corresponding $\widehat{P}$-matrix elements can be easily calculated from their definition (3):

$$\widehat{P}_{1a} = 2d_a p_a \qquad 1 \leqslant a \leqslant 5$$
$$\widehat{P}_{22} = 4p_2$$
$$\widehat{P}_{23} = 0$$

Table 1. Relations defining strata of dimension <4 in the orbit space of $(O(3), \mathbb{R}^8)$.

Stratum	Equalities	Inequalities
$S^{(1)}$	$p_2 = p_4 = p_5 = 12p_1{}^3 - p_3{}^2 = 0$	$p_1 > 0$
$S_A^{(2)}$	$p_1 - p_2 - \frac{p_4{}^2}{12p_2{}^2} = p_3 - \frac{p_4{}^3}{12p_2{}^3} = p_5 - \frac{p_4{}^2}{3p_2} = 0$	$p_2 > 0$
$S_B^{(2)}$	$p_2 = p_4 = p_5 = 0$	$12p_1{}^3 - p_3{}^2 > 0$
$S^{(3)}$	$p_3 - p_4\left(3 - \frac{3p_1}{p_2} + \frac{p_4{}^2}{3p_2{}^3}\right) = p_5 - \frac{p_4{}^2}{3p_2} = 0$	$p_1 > p_2 > 0,$
		$12p_1 p_2{}^2 - 12p_2{}^3 - p_4{}^2 > 0$

$$\widehat{P}_{24} = 4p_4$$
$$\widehat{P}_{25} = 4p_5$$
$$\widehat{P}_{33} = 108(p_1 - p_2)^2$$
$$\widehat{P}_{34} = 18\left(-2p_1 p_2 + 2p_2{}^2 + p_5\right)$$
$$\widehat{P}_{35} = 12(p_2 p_3 + p_1 p_4 - p_2 p_4)$$
$$\widehat{P}_{44} = 12\left(p_2{}^2 + p_5\right)$$
$$\widehat{P}_{45} = 4(p_2 p_3 + 3p_1 p_4 - p_2 p_4)$$
$$\widehat{P}_{55} = \tfrac{4}{3}\left(p_3 p_4 + p_4{}^2 + 9p_1 p_5\right) \tag{23}$$

where the ordered set $(d_1, d_2, d_3, d_4, d_5) = (2, 2, 3, 3, 4)$ specifies the degrees of the polynomials of the MIB.

The determinant of the matrix $\widehat{P}(p)$ factorizes and only one of the two real irreducible factors, that we shall call $A(p)$, turns out to be *active* [24], that is, to be relevant in the determination of the boundary of the orbit space $\mathbb{R}^8/G$:
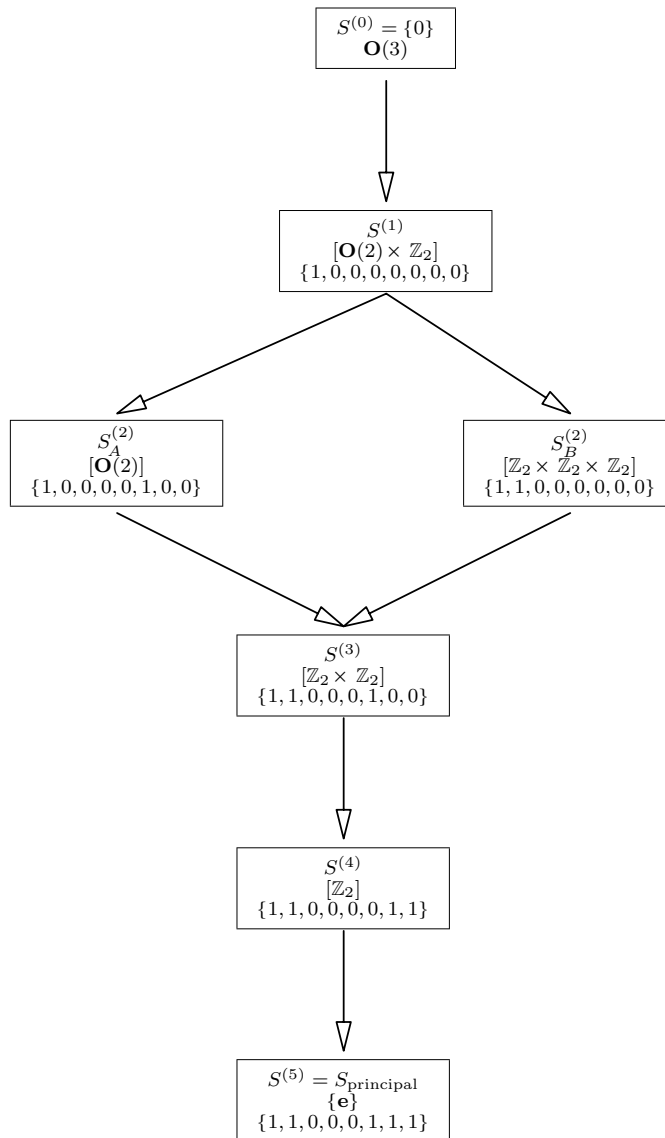
$$\begin{aligned} A(p) = &\ 3p_2{}^3 p_3{}^2 + 18p_1 p_2{}^2 p_3 p_4 - 18p_2{}^3 p_3 p_4 + 27p_1{}^2 p_2 p_4{}^2 - 54p_1 p_2{}^2 p_4{}^2 + 27p_2{}^3 p_4{}^2 \\ &+ p_3 p_4{}^3 - 9p_2 p_3 p_4 p_5 - 9p_1 p_4{}^2 p_5 + 9p_2 p_4{}^2 p_5 - 27p_1 p_2 p_5{}^2 \\ &+ 27p_2{}^2 p_5{}^2 + 9p_5{}^3. \end{aligned} \tag{24}$$

The relations defining strata of dimension <4 in the orbit space of $(O(3), \mathbb{R}^8)$ are summarized in table 1. The isotropy subgroup lattice, with the possible phase transitions between bordering strata, is shown in figure 1.

4.1. Parametrization of the strata

The relations assuring that $\widehat{P}(p) \geqslant 0$ and has rank 4, define a unique four-dimensional stratum $\widehat{S}^{(4)}$ in the orbit space. Using well-known matrix theory results, these conditions could be explicitly written, for instance, in the form $A(p) = 0$ and $M_i(p) > 0$, $i = 1, \ldots, 4$, where M_i is the sum of the principal minors of order i of the matrix $\widehat{P}(p)$: a cumbersome set of conditions that it is not worthwhile to write explicitly.

Even in this simple example one immediately realizes that the choice of a minimal set of explicit algebraic relations providing a cylindrical decomposition [25] for the semialgebraic subset $\widehat{S}^{(4)}$ of $\mathbb{R}^5$ would be a really hard task (for the more peripheral strata, instead, the problem is much easier to solve). An immediate application of the results proved in the previous section, in contrast, leads to a simple rational global parametrization of each stratum, as shown below.

$$\boxed{\begin{array}{c} S^{(0)} = \{0\} \\ \mathbf{O}(3) \end{array}}$$

$$\boxed{\begin{array}{c} S^{(1)} \\ [\mathbf{O}(2) \times \mathbb{Z}_2] \\ \{1, 0, 0, 0, 0, 0, 0, 0\} \end{array}}$$

$$\boxed{\begin{array}{c} S_A^{(2)} \\ [\mathbf{O}(2)] \\ \{1, 0, 0, 0, 0, 1, 0, 0\} \end{array}} \qquad \boxed{\begin{array}{c} S_B^{(2)} \\ [\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2] \\ \{1, 1, 0, 0, 0, 0, 0, 0\} \end{array}}$$

$$\boxed{\begin{array}{c} S^{(3)} \\ [\mathbb{Z}_2 \times \mathbb{Z}_2] \\ \{1, 1, 0, 0, 0, 1, 0, 0\} \end{array}}$$

$$\boxed{\begin{array}{c} S^{(4)} \\ [\mathbb{Z}_2] \\ \{1, 1, 0, 0, 0, 0, 1, 1\} \end{array}}$$

$$\boxed{\begin{array}{c} S^{(5)} = S_{\text{principal}} \\ \{\mathbf{e}\} \\ \{1, 1, 0, 0, 0, 1, 1, 1\} \end{array}}$$

Figure 1. Possible phase transitions between bordering strata are connected by continuous sequences of one or more arrows. For each possible structural phase (isotropy type stratum), a 'typical' point in the order parameter space and the corresponding isotropy subgroup of O(3) are indicated.

4.1.1. Stratum $\widehat{S}^{(4)}$. A 'typical point' of the stratum is $x_{\mathrm{t}} = (1, 1, 0, 0, 0, 0, 1, 1)$. The corresponding isotropy subgroup H of G is the $\mathbb{Z}_2$ group generated by the reflection representing the element diag$(-1, 1, 1) \in \mathrm{O}(3)$ in $\mathbb{R}^8$:

$$\text{diag}(1, 1, -1, -1, 1, -1, 1, 1) \in G.$$

The vector space V formed by the H-invariant vectors of $\mathbb{R}^8$ turns out to be five-dimensional:

$$V = \{x \in \mathbb{R}^8 \mid x_3 = x_4 = x_6 = 0\}.$$

The elements of O(3), corresponding to elements of $\text{Stab}(H, G)$, are block-diagonal matrices of the form $\text{diag}(\pm 1, O)$, with $O \in O(2)$. Therefore, $\text{Stab}(H, G)$ is isomorphic to a group $\mathbb{Z}_2 \times O(2)$ and the quotient group $K = \text{Stab}(H, G)/H$ is the representation in V (induced by the representation G of O(3)) of the $O_1(2)$ subgroup of O(3), formed by the rotations around the first axis.

Using coordinates $v = (x_1, x_2, x_5, x_7, x_8)$ for a vector $v \in V$, the elements of K turn out to be block-diagonal matrices leaving invariant the following subspaces V^A, $1 \leqslant A \leqslant 3$:

$$
\begin{aligned}
V^1 &= \{(x_1, 0, 0, 0, 0) \in V \mid x_1 \in \mathbb{R}\} \\
V^2 &= \{(0, x_2, x_5, 0, 0) \in V \mid x_2, x_5 \in \mathbb{R}\} \\
V^3 &= \{(0, 0, 0, x_7, x_8) \in V \mid x_7, x_8 \in \mathbb{R}\}.
\end{aligned}
\tag{25}
$$

A proper rotation $r(\phi) \in O_1(2)$ of an angle ϕ and the reflection $\text{diag}(-1, 1) \in O_1(2)$ are represented in V by the 5×5 matrices $1 \oplus r(-2\phi) \oplus r(\phi)$ and $\text{diag}(1, -1, 1, -1, 1)$.

Noting that the complex variables $z_1 = x_2 + ix_5$ and $z_2 = x_7 + ix_8$ transform into $\exp(-2i\phi)z_1$ and, respectively, $\exp(i\phi)z_2$ under a rotation and into z_1^* and, respectively, $-z_2^*$ under a reflection, it is easy to realize that a possible MIB for (K, V) is the following:

$$
\begin{aligned}
\lambda_1 &= x_1 \\
\lambda_2 &= |z_1|^2 = x_2^2 + x_5^2 \\
\lambda_3 &= |z_2|^2 = x_7^2 + x_8^2 \\
\lambda_4 &= 2\,\text{Re}\left(z_1 z_2^2\right) = 2\left(x_2 x_7^2 - 2x_5 x_7 x_8 - x_2 x_8^2\right).
\end{aligned}
\tag{26}
$$

It is now easy to express the p in terms of the λ, $p = \phi(\lambda)$, and to check that the following expressions, obtained in this way, identically satisfy the equation $A(p(\lambda)) = 0$ (see (24)) entering the definition of the stratum $\widehat{S}^{(4)}$:

$$
\begin{aligned}
\phi_1(\lambda) &= \lambda_1{}^2 + \lambda_2 + \lambda_3 \\
\phi_2(\lambda) &= \lambda_3 \\
\phi_3(\lambda) &= -2\sqrt{3}\lambda_1\left(\lambda_1{}^2 - 3\lambda_2\right) \\
\phi_4(\lambda) &= \tfrac{\sqrt{3}}{2}(2\lambda_1\lambda_3 - \sqrt{3}\lambda_4) \\
\phi_5(\lambda) &= \lambda_1{}^2\lambda_3 + 3\lambda_2\lambda_3 - \sqrt{3}\lambda_1\lambda_4.
\end{aligned}
\tag{27}
$$

As explained in the previous section, since the group K is coregular, the range Δ for λ is the region where the $\widehat{P}$-matrix $\widehat{\Lambda}(\lambda)$ associated with the MIB $\{\lambda\}$ is positive definite and the rank of the Jacobian matrix $J(\lambda)$ of the transformation $\phi(\lambda)$ is maximum (=4). The matrices $\widehat{\Lambda}(\lambda)$ and $J(\lambda)$ are easily calculated to be

$$
\widehat{\Lambda}(\lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 4\lambda_2 & 0 & 2\lambda_4 \\
0 & 0 & 4\lambda_3 & 4\lambda_4 \\
0 & 2\lambda_4 & 4\lambda_4 & 4\left(\lambda_3{}^2 + 4\lambda_2\lambda_3\right)
\end{pmatrix}
\tag{28}
$$

and

$$
J(\lambda) = \begin{pmatrix}
2\lambda_1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
-6\sqrt{3}\left(\lambda_1{}^2 - \lambda_2\right) & 6\sqrt{3}\lambda_1 & 0 & 0 \\
\sqrt{3}\lambda_3 & 0 & -\sqrt{3}\lambda_1 & -3/2 \\
2\lambda_1\lambda_3 - \sqrt{3}\lambda_4 & 3\lambda_3 & \lambda_1{}^2 + 3\lambda_2 & -\sqrt{3}\lambda_1
\end{pmatrix}.
\tag{29}
$$

Table 2. Isotropy type stratification of the orbit space V/K, whose principal stratum is diffeomorphic to the subprincipal stratum $\widehat{S}^{(4)}$ of the orbit space $\mathbb{R}^8/G$.

Stratum	Equalities	Inequalities
$\Sigma^{(2)}$	$\lambda_3 = \lambda_4 = 0$	$\lambda_2 > 0$
$\Sigma^{(3)}$	$4\lambda_2\lambda_3{}^2 = \lambda_4{}^2$	$\lambda_3 > 0$ and $\lambda_2 \geqslant 0$
Σ_p		$\lambda_2 > 0$ and $\lambda_3 > 0$ and $4\lambda_2\lambda_3{}^2 - \lambda_4{}^2 > 0$

The conditions assuring the positivity of $\widehat{\Lambda}(\lambda) > 0$ can be written in the form

$$\Delta = \left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^4 \mid \lambda_2 > 0 \text{ and } \lambda_3 > 0 \text{ and } \lambda_4^2 < 4\lambda_2\lambda_3{}^2 \right\} \tag{30}$$

and the rank of $J(\lambda)$ turns out to be equal to 4 for all $\lambda \in \Delta$. So we can conclude that the stratum is connected, a piece of information that would be difficult to derive directly from the relations defining the stratum in implicit form (i.e. $A(p) = 0$ and $M_i(p) > 0$).

The parametrization obtained for the sub-peripheral stratum $\widehat{S}^{(4)}$ turns out to be useful, also, because bordering values of λ immediately determine bordering values of p, corresponding to more peripheral strata of $\mathbb{R}^8/G$. The stratification of V/K is summarized in table 2. We would like to remark, however, that there is not a one-to-one correspondence between singular strata of (K, V) corresponding to bordering values of λ and singular strata of $\mathbb{R}^n/G$, corresponding to the associated values of $p(\lambda)$. In fact, as was already noted, since the action of the stabilizer on the subspace V^1 is trivial, the invariant λ_1 has degree 1, so it does not participate in the conditions defining the stratification of (K, V). That is why the parametrization procedure has to be applied *stratum per stratum*.

4.1.2. Stratum $\widehat{S}^{(3)}$. A typical point in this stratum is $x_t = (1, 1, 0, 0, 0, 1, 0, 0)$ and the isotropy subgroup at x_t is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group represented by the O(3) elements $\mathrm{diag}(1, \pm1, \pm1)$. The vector space left invariant by H reduces to the three-dimensional space V:

$$V = \{ x \in \mathbb{R}^8 \mid x_3 = x_4 = x_5 = x_7 = x_8 = 0 \}. \tag{31}$$

The linear group K, acting in V, is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group, generated by the matrices $\mathrm{diag}(1, -1, 1)$ and $\mathrm{diag}(1, 1, -1)$. A MIB for $\mathrm{Stab}(K, V)$ is

$$(\lambda_1, \lambda_2, \lambda_3) = \left(x_1, x_2{}^2, x_6{}^2 \right). \tag{32}$$

The associated matrix $\widehat{\Lambda}(\lambda)$ turns out to be

$$\widehat{\Lambda}(\lambda) = \mathrm{diag}(1, 4\lambda_2, 4\lambda_3). \tag{33}$$

Parametric equations, $p = \phi(\lambda)$, for the stratum can be written in the following form:

$$\begin{aligned}
\phi_1(\lambda) &= \lambda_1{}^2 + \lambda_2 + \lambda_3 \\
\phi_2(\lambda) &= \lambda_3 \\
\phi_3(\lambda) &= -2\sqrt{3}\lambda_1 \left(\lambda_1{}^2 - 3\lambda_2 \right) \\
\phi_4(\lambda) &= -2\sqrt{3}\lambda_1\lambda_3 \\
\phi_5(\lambda) &= 4\lambda_1{}^2\lambda_3
\end{aligned} \tag{34}$$

for λ in the range $\Delta = \{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \mid \lambda_3 > 0 \text{ and } \lambda_2 > 0 \}$.

The rank of the Jacobian matrix $J(\lambda)$ turns out to be equal to 3 for all $\lambda \in \Delta$, so that the stratum is connected.

4.1.3. Stratum $S_A^{(2)}$.   A typical point is $x_t = (1, 0, 0, 0, 0, 1, 0, 0)$ and the isotropy subgroup H of G at x_t is the subgroup formed by the rotations (proper and improper) around the first axis, represented by block-diagonal matrices of the form $\mathrm{diag}(1, O)$, with $O \in O(2)$. The vector space V left invariant by H is two-dimensional:

$$V = \{x \in \mathbb{R}^8 \mid x_2 = x_3 = x_4 = x_5 = x_7 = x_8 = 0\}.$$

K is the $\mathbb{Z}_2$ group generated by the matrix $\mathrm{diag}(1, -1)$. A MIB for (K, V) is the following:

$$(\lambda_1, \lambda_2) = \left(x_1, x_6{}^2\right). \tag{35}$$

The associated matrix $\widehat{\Lambda}(\lambda)$ turns out to be

$$\widehat{\Lambda}(\lambda) = \mathrm{diag}(1, 4\lambda_2). \tag{36}$$

Parametric equations, $p = \phi(\lambda)$, for the stratum can be written in the following form:

$$\begin{aligned}
\phi_1(\lambda) &= \lambda_1{}^2 + \lambda_2 \\
\phi_2(\lambda) &= \lambda_2 \\
\phi_3(\lambda) &= -2\sqrt{3}\lambda_1{}^3 \\
\phi_4(\lambda) &= -2\sqrt{3}\lambda_1\lambda_2 \\
\phi_5(\lambda) &= 4\lambda_1{}^2\lambda_2
\end{aligned} \tag{37}$$

for λ in the range $\Delta = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_2 > 0\}$. The rank of $J(\lambda)$ is equal to 2 on the whole Δ.

4.1.4. Stratum $S_B^{(2)}$.   A typical point is $x_t = (1, 1, 0, 0, 0, 0, 0, 0)$ and the isotropy subgroup H of G at x_t is the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ formed by the reflections of the axes in $\mathbb{R}^3$. The vector space left invariant by H reduces to

$$V = \{x \in \mathbb{R}^8 \mid x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\}.$$

The group K is the finite group of order 6 generated by reflections with root system of type A_2 [23] with generators

$$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}.$$

A MIB for $\mathrm{Stab}(H, G)$ is

$$(\lambda_1, \lambda_2) = \left(x_1{}^2 + x_2{}^2, x_1\left(x_1{}^2 - 3x_2{}^2\right)\right). \tag{38}$$

The associated matrix $\widehat{\Lambda}(\lambda)$ turns out to be

$$\widehat{\Lambda}(\lambda) = \begin{pmatrix} 4\lambda_1 & 6\lambda_2 \\ 6\lambda_2 & 9\lambda_1{}^2 \end{pmatrix}. \tag{39}$$

Parametric equations, $p = \phi(\lambda)$, for the stratum can be written in the following form:

$$(\phi_1(\lambda), \phi_2(\lambda), \phi_3(\lambda), \phi_4(\lambda), \phi_5(\lambda)) = (\lambda_1, 0, -2\sqrt{3}\lambda_2, 0, 0) \tag{40}$$

for λ in the range $\Delta = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 > 0 \text{ and } \lambda_1{}^3 - \lambda_2{}^2 > 0\}$. The rank of $J(\lambda)$ is equal to 2 for all $\lambda \in \Delta$.

Table 3. Strata and isotropy types of the space (SO(3), $\mathbb{R}^8$). For each orbit type, a typical point and the corresponding isotropy subgroup (or its generators) are specified, with reference to the form of the matrices O appearing in (21); $O_\pm$ denote elements of O(2) with determinant ± 1, respectively.

Stratum	Typical point	Isotropy subgroup or set of generators
$S^{(5)}$	$(1, 1, 0, 0, 0, 0, 1, 1, 1)$	$\{e\}$
$S^{(3)}$	$(1, 1, 0, 0, 0, 0, 1, 0, 0)$	$\{\text{diag}(1, -1, -1)\}$
$S_A^{(2)}$	$(1, 0, 0, 0, 0, 0, 1, 0, 0)$	$\text{diag}(1, O_+)$
$S_B^{(2)}$	$(1, 1, 0, 0, 0, 0, 0, 0, 0)$	$\{\text{diag}(1, -1, -1), \text{diag}(-1, 1, -1)\}$
$S^{(1)}$	$(1, 0, 0, 0, 0, 0, 0, 0, 0)$	$\text{diag}(1, O_+) \cup \text{diag}(-1, O_-)$

4.1.5. Stratum $S^{(1)}$. A typical point is $x_t = (1, 0, 0, 0, 0, 0, 0, 0)$ and the isotropy subgroup H of G at x_t consists of the O(3) block-diagonal matrices of the form $\text{diag}(\pm 1, O)$, with $O \in$ O(2). The vector space left invariant by H reduces to

$$V = \{x \in \mathbb{R}^8 \mid x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0\}$$

the group K is trivial and a MIB for $\text{Stab}(K, V)$ is simply $\lambda_1 = x_1$. The associated matrix $\widehat{\Lambda}(\lambda)$ reduces to 1.

Parametric equations, $p = \phi(\lambda)$, for the stratum can be written in the following form:

$$(\phi_1(\lambda), \phi_2(\lambda), \phi_3(\lambda), \phi_4(\lambda), \phi_5(\lambda)) = \left(\lambda_1{}^2, 0, -2\sqrt{3}\lambda_1{}^3, 0, 0\right) \tag{41}$$

for λ in the range $\Delta = \mathbb{R}$. One can verify that the rank of $J(\lambda)$ diminishes in the allowed range Δ for $\lambda_1 = 0$.

4.2. A physically interesting non-coregular variant of the example

The example presented in the first part of this section has been suggested by a paper of Longa and Trebin [28], devoted to the construction of a phenomenological theory of polar structures in chiral biaxial liquid crystals, through the exploitation of the properties, under transformations of a symmetry group SO(3), of a symmetric and traceless tensor order parameter field Q and of a polar vector field P.
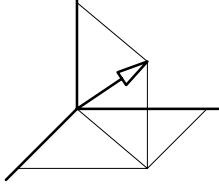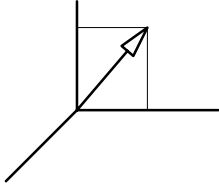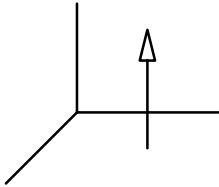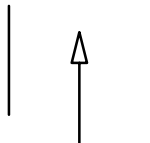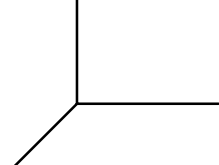
In fact, the situation examined by Longa and Trebin can be recovered from the example worked about, considering the action in $\mathbb{R}^8$ of the subgroup SO(3) of O(3). In this case, we obtain a non-coregular linear group G'. To get a MIB for G', a new G'-invariant polynomial $p_6(x)$, of degree 6, has to be added to the MIB (22):

$$p_6 = 2\sqrt{2}(P \wedge QP) \cdot (Q^2 P). \tag{42}$$

The $\widehat{P}$-matrix associated with the enlarged MIB can be constructed from the one shown in (22) by adding one more row and column:

$$\widehat{P}_{16} = 12p_6$$
$$\widehat{P}_{26} = 6p_6$$
$$\widehat{P}_{36} = 0$$
$$\widehat{P}_{46} = 0 \tag{43}$$
$$\widehat{P}_{56} = 12p_1 p_6$$
$$\widehat{P}_{66} = -\tfrac{1}{9}\left[-p_1{}^2 p_4{}^2 + p_4(p_3 + p_4)p_5 - 3p_1 p_5{}^2 + 2p_1 p_2\left(-p_3 p_4 - p_4{}^2 + 6p_1 p_5\right)\right.$$
$$\left. - p_2{}^2\left(p_3{}^2 - 2p_3 p_4 - 3p_4{}^2 + 12p_1 p_5\right)\right].$$

Table 4. Correspondence between Trebin and Longa's polar states and isotropy type strata for the group O(3); the graphical representation (GR) of the *typical point* is the same as in [28]: the straight lines denote the non-degenerate eigenvector directions of Q, whilst the arrows represent the orientation of the polar vector P with respect to the eigenvectors of Q. The state F_{U2} of [28], for which the angle between the polar vector P and the unique non-degenerate eigenvector of Q is different from 0 and $\pi/2$, actually belongs to the stratum $S^{(4)}$. Therefore, it is written in parenthesis, since it is equivalent (from the symmetry point of view) to the state F_{Bg}.

Stratum	Isotropy subgroup	GR of the *typical point*	Phase in [28]
$S^{(5)}$	$\{e\}$		F_{ch}
$S^{(4)}$	$\{diag(\pm 1, 1, 1)\}$		F_{Bg} (F_{U2})
$S^{(3)}$	$\{diag(1, \pm 1, \pm 1)\}$		$F_{B\parallel}$
$S_A^{(2)}$	$\{diag(1, O), O \in O(2)\}$		F_{U1}
$S_B^{(2)}$	$\{diag(\pm 1, \pm 1, \pm 1)\}$		
$S^{(1)}$	$\{diag(\pm 1, O), O \in O(2)\}$		

The added element $p_6(x)$ is a 'numerator invariant' for the Molien function [29] and is, therefore, algebraically dependent on the other elements of the MIB. The relevant relation can

be easily obtained from one of the irreducible polynomial factors of the enlarged $\widehat{P}$ matrix:

$$243 p_6{}^2 + A(p) = 0 \tag{44}$$

where $A(p)$ is the same (see (24)) as in the coregular case.

G' being a subgroup of G, the lattice of isotropy types of $(\mathrm{SO}(3), \mathbb{R}^8)$ is easily found from the following relation, holding true for all $x \in \mathbb{R}^8$:

$$G'_x = G_x \bigcap G'. \tag{45}$$

An explicit calculation shows that the number of strata of $(\mathrm{SO}(3), \mathbb{R}^8)$ is lower than the number of strata of $(\mathrm{O}(3), \mathbb{R}^8)$. In fact, the four-dimensional stratum $S^{(4)}$ of the second group space is part of the principal stratum of $(\mathrm{SO}(3), \mathbb{R}^8)$. The result is summarized in table 3, where the notation is the same used for the generators in the O(3) case.

As explained at the end of section 2, if SO(3) is assumed to be the (largest) symmetry group of the free energy, the strata of $\mathbb{R}^8/G'$ are in a one-to-one correspondence with the structural phases of the system, i.e., with the phases of the system that are identified only on the basis of their SO(3) symmetry. An easy comparison with the results summarized in table 1 (p 3460) of [28] shows that distinct polar states (F_{ch}, F_{Bg} and F_{U2}) defined by Longa and Trebin lie on the same stratum (the principal stratum) of $(\mathrm{SO}(3), \mathbb{R}^8)$, meaning that their symmetry (orbit-type) is the same: they form a unique structural phase. As stated above, the number of structural phases is increased if the symmetry group of the free energy is enlarged to O(3) (see table 4), but also in this case the polar states F_{Bg} and F_{U2} lie on the same stratum $S^{(4)}$.

In their paper, Longa and Trebin make a big effort to determine the range of the orbit map and produce a classification of the polar states that, however, appears not to be based on their symmetry properties with respect to the symmetry group of the free energy. The effectiveness and mathematical rigour of the orbit space approach to the study of invariant functions clearly emerges from a comparison with the easy calculation, that led us, essentially, to more rigorous results, and with the additional important advantage of keeping, at every step, a clear correspondence between geometrical structures (strata) and symmetry of the corresponding physical configurations. In fact, the parametrization technique becomes very important for orbit spaces of group actions where more than three invariants are involved; here the geometrical intuition based on drawing the shape of the orbit space loses most of its efficiency.

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